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Extensions of the Calderón–Zygmund theory

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Les eines desenvolupades en la dècada de 1950 per Calderón i Zygmund ens permeten demostrar que algunes integrals singulars estan ben definides i fitades en els espais L^p. Tot i que l'espai euclidià fos el context original on totes aquestes idees es varen desenvolupar, aquestes propietats es generalitzen a altres espais mètrics de mesura i a integrals singulars de valors vectorials. Al llarg de les dècades, la teoria ha anat guanyant en abstracció i interès. Encara avui en dia, hi ha operadors que s'escapen de l'abast de la teoria, com és l'operador diàdic esfèric maximal.

Abstract *(ENG)*
The tools developed in the 1950s by Calderón and Zygmund enable us to prove that certain singular integrals are well defined and bounded in L^p spaces. Although the Euclidean space was the original context where all these ideas were developed, these properties generalise to other measure metric spaces and to vector-valued singular integrals. Along the decades, the theory has been acquiring abstraction and luring attention. Even nowadays, there are operators that fall outside the scope of the theory, for instance the dyadic spherical maximal operator.

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1. Introduction

By "singular integral operators" we mean, in the first instance, convolution operators in $\mathbb{R}^n$ the kernel function of which presents a singularity, say, at the origin. Namely, we think of operators of the kind

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy, \quad x \in \mathbb{R}^n,$$

for some given function K that blows up at the origin. Singular integrals show up in a number of problems of analytic nature. For instance, they generate solutions of some partial differential equations, they arise in complex analysis, they underpin apparently unrelated settings in geometric measure theory, etc. See Figure 1 for an illustrative example.

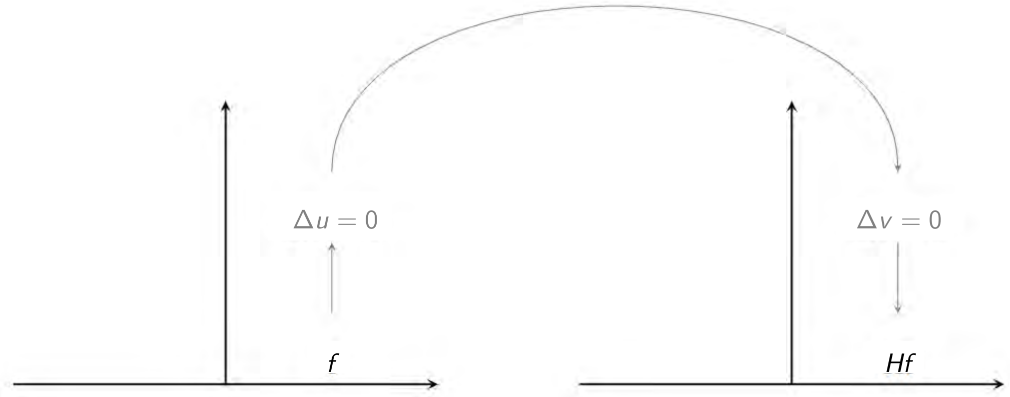

Figure 1: Appearance of the Hilbert transform (the most iconic example of singular integral in $\mathbb{R}$) in Dirichlet's problem for the Laplace equation. First, let f be defined on the axis $y = 0$. Obtain u such that $\Delta u = 0$ in the upper half plane and f is the boundary value of u. Then, get the conjugate harmonic function v of u (the one that turns $u(x,y) + i\,v(x,y)$ into a holomorphic function on the complex plane). Finally, obtain the Hilbert transform of f, Hf, by computing the limit $\lim_{y\to 0} v(x,y)$.

For decades, analysts felt uncomfortable when utilising singular integrals because there was no knowledge regarding their boundedness properties. Were they handling continuous operators on L^p spaces or not? In order to answer this question, Harmonic Analysis is the natural framework.

In the middle and end of the 20^{th} century, the field experienced a burst. Brilliant mathematicians contributed to the expansion of the theory concerning singular integrals. Calderón, Zygmund, Bourgain and Stein are just some of the most influential driving forces in the field, who built upon the work of other great figures like Hardy, Littlewood and Paley.

In the literature, singular integrals are ubiquitous, as they serve to step forward at stages within problems of different natures. Despite this, theory of singular integrals is often just partially explained and treated as an instrument. In this document, we centre them in the spotlight.

2. Calderón–Zygmund theory

The Calderón–Zygmund theory was developed originally in the setting of $\mathbb{R}^n$ in the 1950s, set off by the collaborative breakthrough paper [3] published in 1952. It aimed to prove boundedness of singular convolution-type operators on spaces of functions (mainly L^p spaces) built over $\mathbb{R}^n$.

The starting point is a decomposition lemma that, given an integrable function, enables to split the domain $\mathbb{R}^n$ into a set where the function is bounded, and another set where, although the function may be unbounded, it is controlled in average.

Lemma 2.1 (Calderón–Zygmund lemma in $\mathbb{R}^n$; see [5, Chapter 1, Theorem 4])**.** *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. There exists a partition $\mathbb{R}^n = F \sqcup \Omega$, such that*

(a) $|f(x)| \leq \lambda$ a.e. $x \in F$, and

(b) Ω can be written as a countable union of cubes Q_k with disjoint interior $\Omega = \bigsqcup_{k\in\mathbb{N}} Q_k$, moreover satisfying

$$\lambda \leq \frac{1}{|Q_k|}\int_{Q_k} |f(x)|\,dx \leq 2^n\lambda, \quad \forall\, k \in \mathbb{N}. \tag{1}$$

Proof. Mesh $\mathbb{R}^n$ into a set of cubes $\{Q_k^0\}_{k\in\mathbb{N}}$ with disjoint interiors and of the same size, large enough so that the averages of $|f|$ are bounded above by the given λ on all of the cubes in the mesh:

$$\frac{1}{|Q_k^0|}\int_{Q_k^0} |f(x)|\,dx < \lambda, \quad \forall\, k \in \mathbb{N}.$$

This is possible because f is integrable,

$$\frac{1}{|Q_k^0|}\int_{Q_k^0} |f(x)|\,dx \leq \frac{\|f\|_1}{|Q_k^0|},$$

so choose the size of the cubes such that $|Q_k^0| > \frac{\|f\|_1}{\lambda}$.

We are going to run an algorithm in order to construct F and Ω. Set $\Omega = \emptyset$ and the step $s = 1$. We split each of the cubes Q_k^0 into 2^n dyadic descendant cubes of the same size Q_k^1.

Case 1: For each descendant cube in step s (that is, for each $k \in \mathbb{Z}$), if

$$\frac{1}{|Q_k^s|}\int_{Q_k^s} |f(x)|\,dx > \lambda,$$

then Q_k^s is selected to take part in the set Ω, so update $\Omega^{\text{new}} = \Omega^{\text{old}} \cup Q_k^s$. For such a cube Q_k^s, assume that Q_r^{s-1} is its direct ancestor. Then, by (2) and the fact that Q_r^{s-1} fell into Case 2,

$$\lambda < \frac{1}{|Q_k^s|}\int_{Q_k^s} |f(x)|\,dx \leq \frac{2^n}{|Q_r^{s-1}|}\int_{Q_r^{s-1}} |f(x)|\,dx \leq 2^n\lambda, \tag{2}$$

which proves (1) for Q_k^s.

Case 2: Instead, if

$$\frac{1}{|Q_k^s|}\int_{Q_k^s} |f(x)|\,dx \leq \lambda,$$

then we iterate and further divide Q_k^s into 2^n identical descendant cubes (each with half the sidelength of the ancestor), and check into which of the two cases each of them falls.

Update $s^{\text{new}} = s^{\text{old}} + 1$ and let the algorithm run recursively. This way, we obtain the desired partition $\mathbb{R}^n = F \sqcup \Omega$, Ω being the union of all those cubes that fell into Case 1, and F being the complement of Ω. Plus, (b) has been verified for all cubes Q_k^s that were selected for Case 1. Fact (a) follows from the Lebesgue differentiation theorem: if $x \in F$, this means that there exists a sequence of nested dyadic cubes containing x, $(Q^s_{k(s)})_{s\in\mathbb{N}}$, $(Q^s_{k(s)}) \supset (Q^{s+1}_{k(s+1)})$ being direct dyadic descendants $\forall\, s \in \mathbb{N}$, such that all of these cubes fell into Case 2, implying that

$$f(x) = \lim_{s\to\infty} \frac{1}{|Q^s_{k(s)}|} \int_{Q^s_{k(s)}} |f(y)|\, dy \le \lambda. \qquad \square$$

This decomposition of the domain $\mathbb{R}^n$ of f leads to a useful decomposition of the function f itself. By defining

$$g(x) := \begin{cases} f(x), & x \in F, \\ \dfrac{1}{|Q_k|}\displaystyle\int_{Q_k} f(x)\, dx, & x \in Q_k, \end{cases}$$

and $b(x) := f(x) - g(x)$, we reach the following corollary.

Corollary 2.2 (See [5, Chapter 2, Theorem 1]). *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. There exists a decomposition of f as sum of two functions, $f = g + b$ such that:*

(a) $g(x) \le 2^n\lambda$ *a.e.* $x \in \mathbb{R}^n$,

(b) $\dfrac{1}{|Q_k|}\displaystyle\int_{Q_k} b(x)\, dx = 0\ \forall\, k \in \mathbb{N}$,

(c) $\dfrac{1}{|Q_k|}\displaystyle\int_{Q_k} |b(x)|\, dx \le 2^n\lambda\ \forall\, k \in \mathbb{N}$,

(d) $\operatorname{supp}(b) = \bigsqcup_{k\in\mathbb{N}} Q_k$ *and*

(e) $b \le f$ *a.e.*

The functions g and b are usually referred to as the "good" and the "bad" part of f. Corollary 2.2 is the key ingredient to prove Theorem 2.4, that allows us to bound singular integral operators. However, as one may guess, we first need to make some assumption on the regularity of the singular kernel function. The minimal known hypothesis that succeeds is the so-called Hörmander's condition.

Definition 2.3. A convolution kernel K on $\mathbb{R}^n$ is said to satisfy *Hörmander's condition* if

$$B := \sup_{|y|>0} \int_{|x|\ge 2|y|} |K(x-y) - K(x)|\, dx < \infty. \tag{3}$$

Since the integral is computed over the region $\{x \in \mathbb{R}^n : |x| > 2|y|\}$, the singularity of the kernel is avoided both for $x - y$ and for x. In some sense, we are asking that the global variation of the kernel is not so wild that is not integrable. Nevertheless, Hörmander's condition is usually seen as a weakened version of the stronger condition

$$|\nabla K(x)| \le \frac{C}{|x|^{n+1}},$$

for all $x \in \mathbb{R}^n$ away from the origin. All in all, here is the theorem that gives meaning to the theory. In the literature, one can find many variations and consequences of it.

Theorem 2.4 (See [5, Chapter 2, Sections 2 and 3]). *Let T be a linear operator such that there exists a measurable kernel function K such that*

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

converges absolutely whenever $f \in L^2(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f)$. Suppose the following:

(i) T is bounded on $L^2(\mathbb{R}^n)$: there exists $A > 0$ such that for all $f \in L^2(\mathbb{R}^n)$, $\|Tf\|_2 \leq A\|f\|_2$.

(ii) The kernel K satisfies Hörmander's condition (3) with constant B.

Then,

(a) T is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and

$$\|Tf\|_p \leq C_{n,p}\|f\|_p,$$

for $f \in L^p(\mathbb{R}^n)$ and $C_{n,p} > 0$ only depending on n, p, A and B.

(b) T is weak-type $(1,1)$, i.e., for all $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$,

$$\lambda|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq C_n\|f\|_1,$$

where $C_n > 0$ is a constant only depending on the dimension n, A and B.

The strategy for the proof is, accounting for the boundedness assumption on the Hilbert space $L^2(\mathbb{R}^n)$, using the Calderón–Zygmund lemma to first show (b), i.e., that T is weak-type $(1,1)$. After that, one can use the Marcinkiewicz interpolation theorem between $p = 1$ and $p = 2$ to get (a) for $1 < p \leq 2$. Eventually, a duality argument covers the dual range $2 \leq p < \infty$.

3. Extensions of the theory

In view of Theorem 2.4, it is natural to wonder if it admits generalisations to other settings. Indeed, under suitable conditions, it is possible to extend the theorem, on the one hand, to other measure metric spaces, and on the other hand, to vector-valued functions. The first setting is useful, for example, in the theory of parabolic PDEs, whereas the latter generalisation turns out to be handy to study maximal operators or operators of the kind "square functions". In this section, we present such an abstraction accounting for the combination of both extensions.

Definition 3.1. A measure metric space $((X,d),\Sigma,\mu)$ is said to have the *doubling property* if

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x)), \quad \forall\, r > 0,\ x \in X,$$

$C > 0$ being a universal constant for the space X. This is, measures of dilated balls are comparable.

The doubling property is crucial if we need available inequalities of the kind (2). Along this section, $((X,d),\Sigma,\mu)$ denotes a generic σ-finite measure space over a metric space equipped with a regular measure enjoying the doubling property.

Next, note that in an arbitrary metric space, cubes are not available anymore, but only balls. Therefore, the proof of Lemma 2.1 completely breaks apart, since it relies heavily on meshing $\mathbb{R}^n$ into cubes. This implies that the strategy to get a lemma of the same flavour has to be totally different. To this end, the Hardy–Littlewood maximal function aids.

Definition 3.2. Let $((X,d),\Sigma,\mu)$ be a measure metric space and let $f \in L^1_{\mathrm{loc}}(X)$ be a locally integrable function. The *centred Hardy–Littlewood maximal function* of f is defined as

$$\mathfrak{M}f(x) := \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, d\mu(y). \tag{4}$$

Similarly, the *uncentred Hardy–Littlewood maximal function* of f reads as

$$\mathfrak{M}^{\mathrm{unc}}f(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls B containing x.

When the doubling property is in force, then the centred and uncentred version are easily checked to be comparable. It is also remarkable to note that the Hardy–Littlewood maximal function defines a bounded operator on L^p spaces, $1 < p < \infty$ ([5, Chapter 1, Theorem 1]). In fact, in order to show L^p-boundedness for a broad class of so-called Calderón–Zygmund operators (those under the hypotheses of Theorem 2.4 or Theorem 3.5), one can first show, as pointed out, that the Hardy–Littlewood maximal function is L^p-bounded, and then use this specific result to prove L^p-boundedness for the broad class of Calderón–Zygmund operators.

Lemma 3.3 (Calderón–Zygmund lemma in the general setting; see [7, Chapter 1, Theorem 2])**.** *Let $f \in L^1(X)$ and $\lambda > 0$. There exists a partition of the space $X = F \sqcup \Omega$, F being a closed set and Ω an open set, such that*

(a) $|f(x)| \leq \lambda$ a.e. $x \in F$, and

(b) Ω can be written as a countable disjoint union of smaller sets $\Omega = \bigsqcup_{k\in\mathbb{N}} \Omega_k$ moreover satisfying

$$\frac{1}{\mu(\Omega_k)} \int_{\Omega_k} |f(x)|\, d\mu(x) \leq C\lambda, \quad \forall\, k \in \mathbb{N},$$

for some constant $C > 0$.

Proof. Let $f \in L^1(X)$ and fix $\lambda > 0$. Choose $F := \{x \in X : \mathfrak{M}f(x) \leq \lambda\}$ and so $\Omega := \{x \in X : \mathfrak{M}f(x) > \lambda\}$, being respectively closed and open, because $\mathfrak{M}f(x)$ is a continuous function of x.

By the Lebesgue differentiation theorem, for a.e. $x \in F$,

$$\lambda \geq \mathfrak{M}f(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, d\mu(y) \geq \lim_{r\to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, d\mu(y) = |f(x)|,$$

so (a) is shown.

Let us introduce some notation. For a ball $B = B_r(x)$ centred at x with radius r and for some universal constants $0 < C^* < C^{**}$, denote by $B^* := B_{C^*r}(x)$ and $B^{**} := B_{C^{**}r}(x)$ the centred dilations by factors C^* and C^{**}, respectively. In order to prove (b), we use a Vitali-type covering lemma ([7, Chapter 1, Lemma 2]): given the closed set F, there exists a sequence of balls $(B_k)_{k\in\mathbb{N}}$ and two families of each dilations (or universal dilation constants $0 < C^* < C^{**}$), $(B_k^*)_{k\in\mathbb{N}}$ and $(B_k^{**})_{k\in\mathbb{N}}$, such that

(a) $(B_k)_{k\in\mathbb{N}}$ are pairwise disjoint,

(b) $\bigcup_k B_k^* = F^c$, and

(c) $B_k^{**} \cap F \neq \emptyset$, $\forall\, k$.

It is convenient to extract another sequence of sets. Take the first element in $(B_k^*)_{k\in\mathbb{N}}$ and define $Q_1 := B_1^*$. Next, define $Q_2 := B_2^* \setminus (Q_1)$. By an inductive process, build

$$Q_k := B_k^* \setminus \left(\bigcup_{j=1}^{k-1} Q_j \right).$$

It is directly deduced that the sets Q_k satisfy $\bigcup_k Q_k = F^c$ just like the B_k^*, although with the advantage that the Q_k are pairwise disjoint. The downside, compared to the B_k^*, is that the Q_k are no longer balls, but other less elementary sets. The name Q_k of such new sets is inspired by their role in the proof of Theorem 3.5, which mimics the one carried out by the cubes in the proof of the $X = \mathbb{R}^n$ case.

Now, for each B_k in the sequence $(B_k)_{k\in\mathbb{N}}$, choose a point $p_k \in B_k^{**} \cap F$. By the definition of F,

$$\begin{aligned} \lambda &\geq \mathfrak{M}f(p_k) \geq C^{\mathrm{unc}}\mathfrak{M}^{\mathrm{unc}}f(p_k) \geq \frac{C^{\mathrm{unc}}}{\mu(B_k^{**})} \int_{B_k^{**}} |f(x)|\, d\mu(x) \\ &\geq \frac{C^{\mathrm{unc}}}{\mu(B_k^{**})} \int_{Q_k} |f(x)|\, d\mu(x) \geq \frac{C^{\mathrm{unc}}}{C^{\mathrm{dp}}} \frac{1}{\mu(Q_k)} \int_{Q_k} |f(x)|\, d\mu(x), \end{aligned} \tag{5}$$

where C^{dp} is the constant from the doubling property (see Definition 3.1) and C^{unc} is the constant in the equivalence

$$\mathfrak{M}f \leq \mathfrak{M}^{\mathrm{unc}}f \leq C^{\mathrm{unc}}\,\mathfrak{M}f.$$

In fact, $C^{\mathrm{unc}} = (C^{\mathrm{dp}})^{-1}$. The two last inequalities in (5) stem from the fact that $B_k \subseteq Q_k \subseteq B_k^{**}$ and the doubling property: $\mu(Q_k) \leq \mu(B_k^{**}) \leq C^{\mathrm{dp}}\mu(B_k) \leq C^{\mathrm{dp}}\mu(Q_k)$. Since $(Q_k)_{k\in\mathbb{N}}$ partition Ω, $\Omega = \bigsqcup_k \Omega_k \equiv \bigsqcup_k Q_k$, the proof is complete. □

Note that this proof unveils the precise identity of the sets F and Ω, which are defined in terms of the Hardy–Littlewood maximal function.

In exactly the same way as in Corollary 2.2, the Calderón–Zygmund decomposition of an integrable function $f \in L^1(X)$ as $f = g + b$ is deduced.

We mentioned that we wish our generalised theorem to hold for vector-valued functions. The construction of the L^p spaces for such functions is nowadays standard ([4, Chapter 5]). Let us denote by $L^p_B(X)$ the Lebesgue space of L^p-integrable functions on some measure space X and taking values in the Banach space B. This is, for $1 \leq p < \infty$, set

$$L^p_B(X) := \left\{ F\colon X \to B : \int_X \|F(x)\|_B^p \, d\mu(x) < \infty \right\},$$

whereas for $p = \infty$,

$$L^\infty_B(X) := \left\{ F\colon X \to B : \operatorname*{ess\,sup}_{x\in X} \|F(x)\|_B < \infty \right\}.$$

Additionally, denote by $\mathcal{L}(A, B)$ the Banach space of all linear and continuous maps between Banach spaces A and B.

Note that what has been presented so far in this section also applies to Banach-valued functions.

In order not to scatter away from the theory, we need to upgrade Hörmander's condition on kernel functions as follows. In particular, note that the kernel is no longer a function, but rather a linear operator between Banach spaces.

Definition 3.4. Let A and B be Banach spaces. An operator kernel K on the product measure space $((X,d),\Sigma,\mu)\times((X,d),\Sigma,\mu)$ taking values in $\mathcal{L}(A,B)$ is said to satisfy *Hörmander's condition* if

$$D := \sup_{y,y_0\in X}\int_{d(x,y)\geq Cd(y,y_0)} \|K(x,y)-K(x,y_0)\|_{\mathcal{L}(A,B)}\,d\mu(x) < \infty, \tag{6}$$

for some constant $C>1$.

Another important remark is that now, the kernel operator involves two entries instead of just one, compared to the convolution operators. The reason for this is that "$x-y$" does not make sense in general measure metric spaces, since they lack the vector space structure. Thus, we get around this issue by inputting two variables $x\in X$ and $y\in X$, with the understanding that the kernel is singular around $x=y$.

Astonishingly, the natural generalisation of Theorem 2.4 turns out to work in this setting as well!

Theorem 3.5 (See [7, Chapter 1, Theorem 3] and [4, Chapter 5, Theorem 3.4])**.** *Let $((X,d),\Sigma,\mu)$ be a measure metric space with the doubling property. Let A, B be Banach spaces and let T be a linear operator which is represented by*

$$TF(x) = \int_X K(x,y)F(y)\,d\mu(y),$$

whenever $F\in L^\infty_A(X)$ with compact support and $x\notin \operatorname{supp}(F)$, where the vector-valued kernel $K\in\mathcal{L}(A,B)$ is measurable in $X\times X$ and locally integrable away from the diagonal. Assume that

(i) T is bounded from $L^q_A(X)$ to $L^q_B(X)$ for a fixed $1<q\leq\infty$: there exists $C_q>0$ such that for all $F\in L^q_A(X)$, $\|TF\|_{L^q_B(X)}\leq C_q\|F\|_{L^q_A(X)}$, and

(ii) the operator kernel K satisfies Hörmander's condition in (6) with constants C and D.

Then,

(a) the operator T has a bounded extension mapping $L^p_A(X)$ to $L^p_B(X)$, with $1<p<q$. Furthermore,

$$\|TF\|_{L^p_B(X)}\leq C_p\|F\|_{L^p_A(X)},\quad 1<p<q,$$

for $F\in L^p_A(X)$ and $C_p>0$ only depending on p, q, C_q, C and D.

(b) The operator T has a bounded weak-type $(1,1)$ extension that satisfies

$$\lambda\,\mu(\{x\in X:\|TF(x)\|_B>\lambda\})\leq C_1\|F\|_{L^1_A(X)},\quad\forall\,\lambda>0,$$

for $F\in L^1_A(X)$ and $C_1>0$ only depending on q, C_q, C and D.

The proof follows the strategy of that of Theorem 2.4, just this time using Lemma 3.3 instead of Lemma 2.1, and caring about the technical details of working in the general case.

Here is an example of operator that falls under the scope of the theory.

Example 3.6 (Smooth Littlewood–Paley square function)**.** Let P_j be smooth Littlewood–Paley projectors. Namely, the P_j are defined as multipliers on the Fourier side:

$$\widehat{P_jf}(\xi) := \psi(2^{-j}\xi)\hat{f}(\xi),\quad\forall j\in\mathbb{Z}.$$

Here, ψ is a smooth compactly supported function, the dyadic dilations of which form a partition of unity in frequency. This way, $P_j f$ captures the "part" of f with frequencies around 2^j.

The operator

$$Sf(x) := \left(\sum_{j\in\mathbb{Z}} |P_j f(x)|^2 \right)^{\frac{1}{2}}$$

is named smooth Littlewood–Paley square function.

First of all, we like to think of the square function as the norm of an operator acting on vector-valued functions $S\colon L^p(\mathbb{R}^n) \to L^p_{\ell^2}(\mathbb{R}^n)$: Define

$$\begin{aligned} P(f) &:= (P_j f)_{j\in\mathbb{Z}} = (\dots, P_{-1}f, P_0 f, P_1 f, \dots) \\ &= (\dots,\ 2^{-n}\check{\psi}(2^{-1}\xi) * f(x),\ 2^0\check{\psi}(2^0\xi) * f(x),\ 2^n\check{\psi}(2^1\xi) * f(x), \dots), \end{aligned}$$

which is a *linear* operator mapping functions to sequences of functions.[1] Accordingly,

$$Sf(x) = \|Pf(x)\|_{\ell^2(\mathbb{Z})}.$$

We brought the square function to the vector-valued setting. At this point, one would attempt to apply Theorem 3.5 to Sf. Nonetheless, a direct application fails to show that Sf is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. It is necessary to combine Theorem 3.5 with a probabilistic trick involving Rademacher random variables to eventually show that Sf is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

4. Beyond the paradigm

Together with the development of the Calderón–Zygmund theory as well as its extensions, new problems arose in the field. In particular, interest was shown in singular measure operators. The reason for this interest relies on the thirst for understanding other appealing problems like the Kakeya problem, the Bochner–Riesz conjecture or the Fourier restriction problem, which still remain mysterious and open. Let us give an example in this direction of an operator that is still not completely understood.

Definition 4.1. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function in $\mathbb{R}^n$. Define the *dyadic spherical maximal function* as

$$\tilde{S}f(x) := \sup_{k\in\mathbb{Z}} \int_{\mathbb{S}^{n-1}} f(x - 2^k\omega)\, d\sigma(\omega), \quad \forall x \in \mathbb{R}^n, \tag{7}$$

where $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the unit sphere, σ is the surface measure of $\mathbb{S}^{n-1}$ and $\omega \in \mathbb{S}^{n-1}$ is a unit vector.

The (non-maximal) spherical means appear in the expression for the solution to the Cauchy problem of the wave equation in odd space dimension. The interest in studying its maximal versions relies on the availability of a standard strategy to prove pointwise convergence results of the solution to the wave equation towards the initial datum.

The operator (7) is similar to the Hardy–Littlewood maximal function (4) in the sense that, instead of averaging over balls, it averages over spheres. However, the surface measure of $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ is a singular

[1] One can play the same trick with maximal functions. For instance, $\mathfrak{M}f(x) = \|A(x,\cdot)f\|_{L^\infty(\mathbb{R}_{>0})}$, where $A(x,r)f$ denotes the average of f on the ball centred at x of radius r.

measure, in the sense that all of its mass is concentrated on a null n-Lebesgue measure manifold. Furthermore, (7) can be seen as a convolution of a function f against a (singular) measure, but not another function anymore. This brings obstacles to our understanding of the spherical maximal function, because the Calderón–Zygmund theory from previous sections does not apply anymore.

In this case, the radii are discretised. It is of course of interest to take supremum over the continuum $r > 0$. In that case, the spherical maximal function has been understood deeply and it turns out that the boundedness properties depend on the dimension [1, 6]. Up to the date, we know that this dyadic version defines indeed a bounded operator on $L^p(\mathbb{R}^n)$. Nonetheless, we do not know whether it is weak-type $(1,1)$.

Theorem 4.2 (See [2])**.** *The dyadic spherical maximal operator $\tilde{\mathcal{S}}f$ is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. This is, for $f \in L^p(\mathbb{R}^n)$,*

$$\|\tilde{\mathcal{S}}f\|_p \leq C_p \|f\|_p,$$

for some constant $C_p > 0$ depending on p and n.

Conjecture 4.3. *The dyadic spherical maximal operator $\tilde{\mathcal{S}}f$ is weak-type $(1,1)$. So for any $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$,*

$$\lambda|\{x \in \mathbb{R}^n : \tilde{\mathcal{S}}f(x) > \lambda\}| \leq C_1 \|f\|_1,$$

for some constant $C_1 > 0$ depending on n.

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